L09 Critical function for a test

1. Randomized test

   (1) Reachable level

   Let the space for parameter θ is partitioned by H_0 and H_a, and the range of test statistic $T(X)$ is partitioned by R_T and A_T. For test

   > $H_0 : \theta \in H_0$ versus $H_a : \theta \in H_a$
   > Test statistic: $T(X)$
   > Reject H_0 if $T(X) \in R_T$

   has $\beta(\theta) = P_\theta(\text{Rejecting } H_0) = P_\theta(X(X) \in R_T)$
   with significance level α: $\beta(\theta) \le \alpha$ for all $\theta \in H_0$.
   This level is reachable if $\exists\, \theta_0 \in H_0$ such that $\beta(\theta_0) = \alpha$.
   If $T(X)$ has discrete distribution, level α may or may not be reached.

   (2) Randomized test

   Suppose the range of test statistic $T(X)$ is partitioned by R_t, P_T and A_T. Then test

   > $H_0 : \theta \in H_0$ versus $H_a : \theta \in H_a$
   > Test statistic: $T(X)$
   > Reject H_0 with probability 1 if $T(X) \in R_T$
   > with probability r if $T(X) \in P_T$
   > with probability 0 if $T(X) \in A_T$

   is a randomized test with $\beta(\theta) = P_\theta(T(X) \in R_t) + r \cdot P(T(X) \in P_T)$.
   For a randomized test one may select R_T, P_T and r to make the level α reachable.

2. Critical function of a test

   (1) For the test in (1) of 1,

   > $H_0 : \theta \in H_0$ versus $H_a : \theta \in H_a$
   > Critical function $\phi(X) = \begin{cases} 1 & T(X) \in R_T \\ 0 & \text{Otherwise} \end{cases}$
   > Reject H_0 with probability $\phi(X)$

   This test has $\beta(\theta) = 1 \cdot P_\theta(T(X) \in R_T) + 0 \cdot P_\theta(T(X) \in R_T^c) = E_\theta[\phi(X)]$.

   (2) For the test in (2) of 1,

   > $H_0 : \theta \in H_0$ versus $H_a : \theta \in H_a$
   > Critical function $\phi(X) = \begin{cases} 1 & T(X) \in R_T \\ r & T(X) \in P_T \\ 0 & T(X) \in A_t \end{cases}$
   > Reject H_0 with probability $\phi(X)$

This test has

$$\begin{aligned}\beta(\theta) &= 1\cdot P_\theta(T(X)\in R_T) + r\cdot P(T(X)\in P_T) + 0\cdot P_\theta(T(X)\in A_T)\\ &= E_\theta[\phi(X)]\end{aligned}$$

3. A likelihood ratio test

(1) A LRT

Suppose $\theta \in \{\theta_0,\, \theta_1\}$. pdf/pmf $f(X;\,\theta_0) \stackrel{def}{=\!=} f_0(X)$ and $f(X;\,\theta_1) \stackrel{def}{=\!=} f_1(X)$.
Let $\Lambda(X) = \frac{f_1(X)}{f_0(X)}$.

$H_0:\,\theta=\theta_0$ versus $H_a:\,\theta=\theta_1$
Critical function $\phi(X) = \begin{cases} 1 & \Lambda(X) > k \\ r & \Lambda(X) = k \\ 0 & \Lambda(X) < k \end{cases}$
Reject H_0 with probability $\phi(X)$

has $\beta(\theta) = E_\theta[\phi(X)] = P_\theta(\Lambda > k) + r\cdot P_\theta(\Lambda = k)$
with level $\alpha = E_{\theta_0}[\phi(X)] = P_{\theta_0}(\Lambda > k) + r\cdot P_{\theta_0}(\Lambda = k)$.
So $r = \frac{\alpha - P_{\theta_0}(\Lambda > k)}{P_{\theta_0}(\Lambda = k)}$.
One can find k and r to make α reachable.

(2) Equivalent tests

Suppose Λ is an increasing function of T^* and a decreasing function of T_*. Then

$$\Phi(X) = \begin{cases} 1 & \Lambda > k \\ r & \Lambda = k \\ 0 & \Lambda < k \end{cases} = \begin{cases} 1 & T^* > k_1 \\ r & T^* = k_1 \\ 0 & T^* < k_1 \end{cases} = \begin{cases} 1 & T_* < k_2 \\ r & T_* = k_1 \\ 0 & T_* > k_2 \end{cases}$$

$$\begin{aligned}\text{with } \beta(\theta) &= E_\theta[\phi(X)] = P_\theta(\Lambda > k) + r\cdot P_\theta(\Lambda = k)\\ &= P_\theta(T^* > k_1) + r\cdot P_\theta(T^* = k_1) = P_\theta(T_* < k_2) + r\cdot P_\theta(T_* = k_2)\end{aligned}$$

Ex: With Bernoulli(p) design a test on $H_0:\,p = 0.3$ versus $H_a:\,p = 0.5$ at the level $\alpha = 0.05$, based on a sample of size $n = 20$,
$\Lambda = \frac{f(X;\,0.5)}{f(X;\,0.3)} = \frac{(0.5)^{\sum X_i}(0.5)^{20-\sum X_i}}{(0.3)^{\sum X_i}(0.7)^{20-\sum X_i}} = \left(\frac{7}{3}\right)^{\sum X_i}\left(\frac{5}{7}\right)^{20}$
is an increasing function of $T = \sum_{i=1}^{20} X_i \sim B(20,\, p)$.
From $\alpha = P_{\theta_0}(T > k) + r\cdot P_{\theta_0}(T = k)$, $0.05 = P(B(20,\,0.3) > k) + r\cdot P(B(10,\,0.3) = k)$.
So $r = \frac{0.05 - P(B(20,0.3)>k)}{P(B(20,0.3)=k)} = \frac{0.05 - P(B(20,0.3)>9)}{P(B(20,0.3)=9)} = \frac{0.05-0.04796}{0.06536} = 0.0312$. Thus

$H_0:\,p = 0.3$ versus $H_a:\,p = 0.5$
Critical function $\phi(X) = \begin{cases} 1 & \sum_{i=1}^{20} X_i > 9 \\ 0.0312 & \sum_{i=1}^{20} X_i = 9 \\ 0 & \sum_{i=1}^{20} X_i < 9 \end{cases}$
Reject H_0 with probability $\phi(X)$

is a test with level 0.05

L10 Neyman-Pearson Lemma

1. Concept of most powerful test

   (1) Representation of a test
   For testing on hypothesis about θ, critical function $0 \le \psi(X) \le 1$ represents scheme.
   Reject H_0 with probability $\psi(X)$.
   Thus $\beta(\theta) = P_\theta(\text{Rejecting } H_0) = \sum \psi(x) P_\theta(X = x) = E_\theta[\psi(X)]$

   (2) For test on $H_0 : \theta = \theta_0$ vs $H_a : \theta = \theta_1$
   with critical function $\psi(X)$,
   $P(\text{Type I Error}) = E_{\theta_0}[\psi(X)]$, power of the test: $E_{\theta_1}[\psi(X)]$
   Significance level is α if $E_{\theta_0}[\psi(X)] \le \alpha$
   So the collection of all α-level test is

$$\tau = \{0 \le \psi(X) \le 1 : E_{\theta_0}[\psi(X)] \le \alpha]\}$$

   (3) α-level most powerful test
   $\phi(X)$ is α-level most powerful test if
   $E_{\theta_0}[\phi(X)] \le \alpha$ and $E_{\theta_0}[\psi(X)] \le \alpha \Longrightarrow E_{\theta_1}[\psi(X)] \le E_{\theta_1}[\phi(X)]$.

2. Neyman-Pearson Lemma

   For $H_0 : \theta = \theta_0$ versus $H_a : \theta = \theta_1$, there exist k and r such that $\phi(X) = \begin{cases} 1 & f_1 > kf_0 \\ r & f_1 = kf_0 \\ 0 & f_1 < kf_0 \end{cases}$
   with $E_{\theta_0}[\phi(X)] = \alpha$, i.e., $\phi(X)$ is an α-level test.

   (1) If $E_{\theta_0}[\psi(X)] \le \alpha$, then $E_\theta[\psi(X)] \le E_\theta[\phi(X)]$ for all $\theta = \theta_0, \theta_1$ so that $\phi(X)$ is α-level most powerful (MP) test.

   Proof. First, $E_{\theta_0}[\psi(X)] \le \alpha = E_{\theta_0}[\phi(X)]$. Then

$$\begin{aligned}
E_{\theta_1}[\phi(X)] - E_{\theta_1}[\psi(X)] &= E_{\theta_1}[\phi(X) - \psi(X)] = \textstyle\sum_x [\phi(X) - \psi(X)] f_1(X) \\
&= \textstyle\sum_x [\phi(X) - \psi(X)][(f_1 - kf_0) + kf_0] \\
&= \textstyle\sum_x [\phi(X) - \psi(X)](f_1 - kf_0) + k\{E_{\theta_0}[\phi(X)] - E_{\theta_0}[\psi(X)]\} \\
&= \textstyle\sum_x [\phi(X) - \psi(X)](f_1 - kf_0) + k\{\alpha - E_{\theta_0}[\psi(X)]\} \\
&\ge \textstyle\sum_x [\phi(X) - \psi(X)](f_1 - kf_0) \\
&= \textstyle\sum_{f_1 - kf_0 > 0} [1 - \psi(X)](f_1 - kf_0) \\
&\quad + \textstyle\sum_{f_1 - kf_0 = 0} [r - \psi(X)](0) \\
&\quad + \textstyle\sum_{f_1 - kf_0 < 0} [0 - \psi(X)](f_1 - kf_0) \\
&\ge 0.
\end{aligned}$$

   So $E_{\theta_1}[\psi(X)] \le E_{\theta_1}[\phi(X)]$.

   (2) If $E_{\theta_0}[\psi(X)] \le \alpha$ and $E_{\theta_1}[\psi(X)] = E_{\theta_1}[\phi(X)]$, then $E_\theta[\psi(X)] = E_\theta[\phi(X)]$ for all $\theta = \theta_0, \theta_1$ so that $\psi(X)$ is also an α-level MP test, and $\psi(X) = \phi(X)$ on $f_1 - kf_0 \ne 0$.

   Proof. Note that

$$\begin{aligned}
0 &= E_{\theta_1}[\phi(X)] - E_{\theta_1}[\psi(X)] = E_{\theta_1}[\phi(X) - \psi(X)] \\
&= \textstyle\sum_x [\phi(X) - \psi(X)] f_1(X) = \sum_x [\phi(X) - \psi(X)][(f_1 - kf_0) + kf_0] \\
&= \textstyle\sum_x [\phi(X) - \psi(X)](f_1 - kf_0) + k \sum_2 [\phi(X) - \psi(X)] f_0 \\
&= \textstyle\sum_{f_1 - kf_0 > 0} [1 - \psi(X)](f_1 - kf_0) + \sum_{f_1 - kf_0 = 0} [r - \psi(X)](0) \\
&\quad + \textstyle\sum_{f_1 - kf_0 < 0} [0 - \psi(X)](f_1 - kf_0) + k\{\alpha - E_{\theta_0}[\psi(X)]\}
\end{aligned}$$

implies $E_{\theta_0}[\psi(X)] = \alpha$ and $\psi(X) = \begin{cases} 1 & f_1 - kf_0 > 0 \\ 0 & f_1 - kf_0 < 0 \end{cases}$.

So $E_{\theta_0}[\psi(X)] = E_{\theta_0}[\phi(X)]$ and $\psi(X) = \phi(X)$ on $f_1 - kf_0 \neq 0$.

3. Examples

(1) Example 1

67.3.2 on p130

$X = (X_1, ..., X_n)$ is a random sample from $N(\theta, \sigma^2)$ where σ^2 is known. Derive an α-level MP test on $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$.

$$\Lambda = \frac{f_1}{f_0} = \frac{\frac{1}{(2\pi\sigma^2)^{n/2}} \exp[\frac{-1}{2\sigma^2} \sum (X_i - \theta_1)^2]}{\frac{1}{(2\pi\sigma^2)^{n/2}} \exp[\frac{-1}{2\sigma^2} \sum (X_i - \theta_0)^2]} = \exp\{\tfrac{1}{2\sigma^2}[n(\theta_0^2 - \theta_1^2) + 2(\theta_1 - \theta_0)n\overline{X}].$$

Case I: $\theta_0 < \theta_1$. Λ is an increasing function of $\overline{X} \sim N\left(\theta, \frac{\sigma^2}{n}\right)$.

Consider $\phi(X) = \begin{cases} 1 & \overline{X} > c \\ 0 & \overline{X} < c \end{cases}$ with $E_{\theta_0}[\phi(X)] = \alpha$.

$\alpha = E_{\theta_0}[\phi(X)] = P_{\theta_0}(\overline{X} > c) = P(Z > \frac{c-\theta_0}{\sigma/\sqrt{n}}) \iff \frac{c-\theta_0}{\sigma/\sqrt{n}} = Z_\alpha \iff c = \theta_0 + Z_\alpha \frac{\sigma}{\sqrt{n}}$.

Thus $\phi(X) = \begin{cases} 0 & \overline{X} < \theta_0 + Z_\alpha \frac{\sigma}{\sqrt{n}} \\ 1 & \overline{X} > \theta_0 + Z_\alpha \frac{\sigma}{\sqrt{n}} \end{cases}$ is α-level MP test.

Case II: $\theta_0 > \theta_1$. Λ is a decreasing function of $\overline{X} \sim N\left(\theta, \frac{\sigma^2}{n}\right)$.

Consider $\phi(X) = \begin{cases} 1 & \overline{X} < c \\ 0 & \overline{X} > c \end{cases}$. with $E_{\theta_0}[\phi(X)] = \alpha$......the rest is skipped.

(2) Example 2

For $H_0 : \theta = \theta_0$ versus $H_a : \theta = \theta_1$ with

X	1	2	3	4
$f(x;\theta_1)$	0.05	0.12	0.16	0.67
$f(x;\theta_0)$	0.01	0.03	0.04	0.92
$\Lambda = \frac{f_1}{f_0}$	5	4	4	0.73

,

find MP test at level $\alpha = 0.05$.

Consider $\phi(X) = \begin{cases} 1 & \Lambda > 4 \\ r & \Lambda = 4 \\ 0 & \Lambda < 4 \end{cases}$. $0.05 = E_{\theta_0}[\phi(X)] = 0.01 + 0.07r \Longrightarrow r = \frac{4}{7}$.

So $\phi(X) = \begin{cases} 1 & \Lambda > 4 \\ \frac{4}{7} & \Lambda = 4 \\ 0 & \Lambda < 4 \end{cases}$ is MP test at level 0.05.

Note that for $\psi(X) = \begin{cases} 1 & \Lambda > 4 \\ \frac{2}{3} & X = 2 \text{ a case of } \Lambda = 4 \\ \frac{1}{2} & X = 3 \text{ a case of } \Lambda = 4 \\ 0 & \Lambda < 4 \end{cases}$,

$E_\theta[\psi(X)] = E_\theta[\phi(X)]$ for all $\theta = \theta_0, \theta_1$.

Thus $\phi(X)$ is also a MP test at level 0.05. Clearly $\psi(x) = \phi(x)$ when $\Lambda \neq 4$.